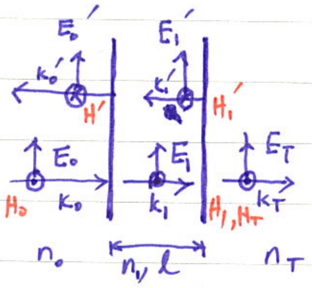
4.4 Theory of Multilayer flim



· Boundary Conditions : continuity at the interface

(i) 1st interface

$$① - E_0 + E_0' = E_1 + E_1'$$

$$- H_0 - H_0' = H_1 - H_1'$$

(ii) 2nd interface

$$E_1 e^{ikl} + E_1' e^{-ikl} = E_T \quad ②$$

$$H_1 e^{ikl} - H_1' e^{ikl} = H_T$$

Using $\bar{H} = \bar{k} \times \bar{E}$ & $k = \frac{2\pi n}{\lambda_n}$ (see 2-7),

$$③ \quad n_0 E_0 - n_0 E_0' = n_1 E_1 - n_1 E_1'$$

$$n_1 E_1 e^{ikl} - n_1 E_1' e^{-ikl} = n_T E_T \quad ④$$

(i) To eliminate E_1', ② × n_1 + ④ :

$$n_1 E_1 e^{ikl} + n_1 E_1' e^{-ikl} = n_1 E_T$$

$$+) \; n_1 E_1 e^{ikl} - n_1 E_1' e^{-ikl} = n_T E_T$$

$$2 n_1 E_1 e^{ikl} = (n_1 + n_T) E_T \rightarrow E_1 = \left(\frac{n_1 + n_T}{2 n_1} \right) E_T e^{-ikl} \quad ⑤$$

(ii) To eliminate E_1, ② × n_1 − ④ :

$$n_1 E_1 e^{ikl} + n_1 E_1' e^{-ikl} = n_1 E_T$$

$$-) \; n_1 E_1 e^{ikl} - n_1 E_1' e^{-ikl} = n_T E_T$$

$$2 n_1 E_1' e^{-ikl} = (n_1 - n_T) E_T \rightarrow E_1' = \left(\frac{n_1 - n_T}{2 n_1} \right) E_T e^{ikl}$$

⑥

(iii) Insert ⑤ & ⑥ into ① :

$$\underline{E_1 + E_1'} = \left(\frac{n_1 + n_T}{2n_1} e^{-ikl} + \frac{n_1 - n_T}{2n_1} e^{ikl} \right) E_T$$

$$= \left[\cos kl - i\left(\frac{n_T}{n_1}\right) \sin kl \right] E_T \quad ⑦$$

$$\underline{= E_0 + E_0'}$$

(iv) Insert ⑤ & ⑥ into ③ :

$$n_1 (E_1 - E_1') = \left(\frac{n_1 + n_T}{2} e^{-ikl} - \frac{n_1 - n_T}{2} e^{ikl} \right) E_T$$

$$= (-i n_1 \sin kl + n_T \cos kl) E_T \quad ⑧$$

$$\underline{= n_0 (E_0 - E_0')}$$

By rewritting ⑦ & ⑧, (divided by E_0) :

$$1 + \frac{E_0'}{E_0} = \left[\cos kl - i\left(\frac{n_T}{n_1}\right) \sin kl \right] \frac{E_T}{E_0} \quad ⑨$$

$$n_0 - n_0 \frac{E_0'}{E_0} = (-i n_1 \sin kl + n_T \cos kl) \frac{E_T}{E_0} \quad ⑩$$

In a matrix form, ⑨ & ⑩ becomes :

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} \frac{E_0'}{E_0} = \begin{bmatrix} \cos kl & \frac{-i}{n_1} \sin kl \\ -i n_1 \sin kl & \cos kl \end{bmatrix} \begin{bmatrix} 1 \\ n_T \end{bmatrix} \frac{E_T}{E_0}$$

or $$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = M \begin{bmatrix} 1 \\ n_T \end{bmatrix} t$$

* Transfer matrix M :

$$M = \begin{bmatrix} \cos kl & -\frac{i}{n_1} \sin kl \\ -i n_1 \sin kl & \cos kl \end{bmatrix}.$$

For N layer interfaces having $n_1, n_2, n_3, \cdots n_N$, and thickness $l_1, l_2, l_3, \cdots l_N$, respectively, the reflection and transmission coefficients of the multilayer film are :

$$\begin{bmatrix} 1 \\ n_0 \end{bmatrix} + \begin{bmatrix} 1 \\ -n_0 \end{bmatrix} r = \underbrace{M_1 M_2 M_3 \cdots M_N}_{M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}} \begin{bmatrix} 1 \\ n_T \end{bmatrix} t$$

$$1 + r = (A + B n_T) t \quad \cdots ①$$

$$n_0 - n_0 r = (C + D n_T) t \quad \cdots ②$$

(i) $n_0 \times ① + ②$:

$$n_0 + n_0 r = n_0 (A + B n_T) t \quad \cdots ①'$$

$$+) \; n_0 - n_0 r = (C + D n_T) t$$

$$2 n_0 = (n_0 A + B n_T n_0 + C + D n_T) t$$

$$\therefore t = \frac{2 n_0}{n_0 A + B n_T n_0 + C + D n_T} \quad (4.31)$$

(ii) $n_0 \times ① - ②$:

$$+2 n_0 r = (n_0 A + B n_T n_0 - C - D n_T) t$$

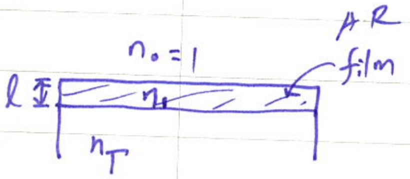
$$\therefore r = \frac{n_0 A + B n_T n_0 - C - D n_T}{n_0 A + B n_T n_0 + C + D n_T} \quad (4.30)$$

(Reflectance $R \, (= |r|^2)$

Transmittance $T \, (= |t|^2)$)

morning glory

< AntiReflecting Films >



$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} \cos kl & -\frac{i}{n_1} \sin kl \\ -i n_1 \sin kl & \cos kl \end{bmatrix} \qquad (4.27)$$

$$r = \frac{A n_0 + B n_T n_0 - C - D n_T}{A n_0 + B n_T n_0 + C + D n_T} \; ; \quad A = D = \cos kl, \quad B = -\frac{i}{n_1} \sin kl, \quad C = -i n_1 \sin kl$$

$$= \frac{(1 - n_T)\cos kl + i\left(n_1 - \frac{n_T}{n_1}\right)\sin kl}{(1 + n_T)\cos kl - i\left(n_1 + \frac{n_T}{n_1}\right)\sin kl}$$

$$= \frac{n_1(1 - n_T)\cos kl - i(n_T - n_1^2)\sin kl}{n_1(1 + n_T)\cos kl - i(n_T + n_1^2)\sin kl} \qquad (4.32)$$

For AR coating, $kl = \pi/2$. $\rightarrow$ $\cos kl = 0$, $\sin kl = 1$

$$r = \frac{n_T - n_1^2}{n_T + n_1^2} \; ; \quad R = \frac{(n_T - n_1^2)^2}{(n_T + n_1^2)^2}$$

$$\rightarrow R = 0 \; ; \quad n_1 = \sqrt{n_T} . \qquad (4.34)$$

ex) MgF : $n_1 = 1.35$

Glass : $n_T \fallingdotseq 1.5$ $\qquad \sqrt{1.5} = 1.22$

$$R = \left(\frac{1.5 - 1.35^2}{1.5 + 1.35^2}\right)^2 = 9.4 \times 10^{-3} \sim 1\%$$

c.f. uncoated glass : $R \sim 4\%$.

See Fig. 4.8

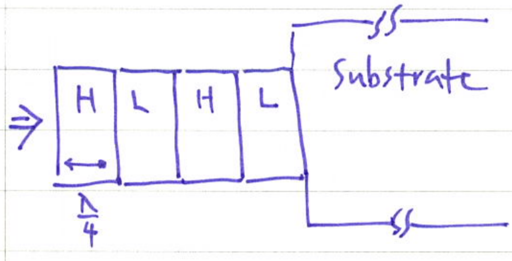
< High Reflection Film >

$$M = \begin{bmatrix} \cos kl & -\frac{i}{n_1}\sin kl \\ -in_1 \sin kl & \cos kl \end{bmatrix}$$

· For $kl = \pi/2$,

$$M = \begin{bmatrix} 0 & -\frac{i}{n_1} \\ -in_1 & 0 \end{bmatrix}$$

ex)



$$M = [M_H][M_L][M_H][M_L]$$

$$= \begin{bmatrix} 0 & -\frac{i}{n_H} \\ -in_H & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{i}{n_L} \\ -in_L & 0 \end{bmatrix} = \begin{bmatrix} -\frac{n_L}{n_H} & 0 \\ 0 & -\frac{n_H}{n_L} \end{bmatrix}$$

if there is $2N$ layers on the substrate,

$$M = \begin{bmatrix} -\frac{n_L}{n_H} & 0 \\ 0 & -\frac{n_H}{n_L} \end{bmatrix}^N = \begin{bmatrix} \left(-\frac{n_L}{n_H}\right)^N & 0 \\ 0 & \left(-\frac{n_H}{n_L}\right)^N \end{bmatrix}$$

$$R \doteq |r|^2 = \left(\frac{An_0 + Bn_Tn_0 - C - Dn_T}{An_0 + Bn_Tn_0 + C + Dn_T}\right)^2 = \left(\frac{(-n_L/n_H)^N - (-n_H/n_L)^N}{(-n_L/n_H)^N + (-n_H/n_L)^N}\right)^2$$

$$= \left[\frac{(n_L/n_H)^{2N} - 1}{(n_L/n_H)^{2N} + 1}\right]^2$$

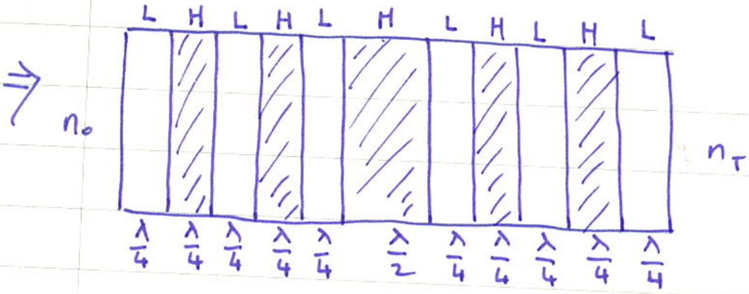
ex) ZnF, $n_H = 2.3$ $\quad \frac{n_L}{n_H} = 0.59 \xrightarrow{N=4} \left(\frac{n_L}{n_H}\right)^8 = 0.014$

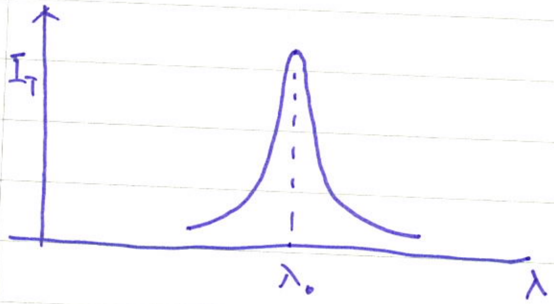
MgF, $n_L = 1.35$

$\therefore R = 0.97$

If $N = 30$, $R = 0.999$

<F-P Interference Filter>





$$M = M_L M_H M_H M_L$$

$$= \begin{bmatrix} 0 & -\frac{i}{n_L} \\ -in_L & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{i}{n_H} \\ -in_H & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{i}{n_H} \\ -in_H & 0 \end{bmatrix} \begin{bmatrix} 0 & -\frac{i}{n_L} \\ -in_L & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{n_H}{n_L} & 0 \\ 0 & -\frac{n_L}{n_H} \end{bmatrix} \begin{bmatrix} -\frac{n_L}{n_H} & 0 \\ 0 & -\frac{n_H}{n_L} \end{bmatrix} = \begin{bmatrix} \frac{n_H}{n_L}\frac{n_L}{n_H} & 0 \\ 0 & \frac{n_L}{n_H}\frac{n_H}{n_L} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{\mathbb{1}}$$

$$R = |r|^2 = \left(\frac{n_0 - n_T}{n_0 + n_T} \right)^2$$ from (4.30)

if $n_0 = n_T = 1$, $R = 0$

$$T = |t|^2 = \frac{2n_0}{n_0 + n_T} = \underline{1}$$